Chapter 2
Geometric Solution of Linear Programming Models

Now that we can set up LP problems, the next step is to learn how to solve them. In this section we will learn how some LP problems can be solved graphically. In order to apply this geometric technique, the LP model must have only two variables, x_1 and x_2. It is possible to geometrically solve problems with three variables but that requires much more sophisticated graphing skills and will not be considered in these notes.

We begin by determining the **feasible region**, which is the set of all points (x_1, x_2) which satisfy the constraint inequalities. We will picture this set using a pair of axes with x_1 being measured along the horizontal axis and x_2 along the vertical axis, as in FIGURE 1.

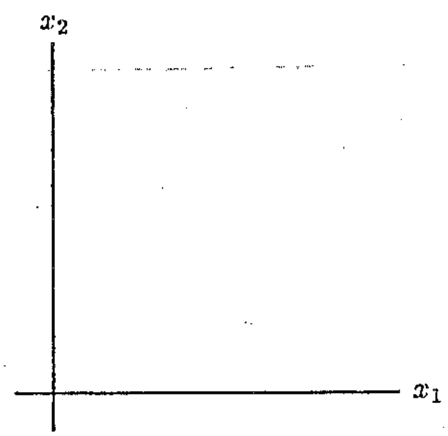


FIGURE 1

A constraint inequality, such as $6x_1 + 7x_2 \leq 8$, can be broken into two parts: $6x_1 + 7x_2 = 8$ and $6x_1 + 7x_2 < 8$. The graph of the equality is a line. For purposes of later generalization we refer to this line in the plane which arises from this equality as a **hyperplane**. All ordered pairs (x_1, x_2) which satisfy the equation part lie on this hyperplane (line).

The second part of the constraint is the strict inequality $6x_1 + 7x_2 < 8$. All ordered pairs of numbers which satisfy this strict inequality make up a **half-space** which lies on one side of the hyperplane (line). To find this half-space, we must decide which side of the hyperplane is the half-space we want. This is done by selecting one point (p, q) (usually the origin $(0,0)$) which does *not* lie on the hyperplane and testing it. The half-space that contains (p, q) is the one we want if (p, q) satisfies the inequality—if $6p + 7q < 8$ is true. The other half-space is the one we want if (p, q) does not satisfy the inequality—if $6p + 7q < 8$ is false. WE CALL (p,q) A TEST POINT – T.P..

EXAMPLE 1: Graphically indicate the solution set of $3x_1 + 2x_2 \le 6$.

Solution: First we sketch the hyperplane (line) whose equation is $3x_1 + 2x_2 = 6$. One way to do this is to plot the two intercepts and draw the line through the two points. If $x_1 = 0$ then $x_2 = 3$. If $x_2 = 0$ then $x_1 = 2$. We plot the two points $(0,3)$ and $(2,0)$ and sketch the line through them in FIGURE 2.

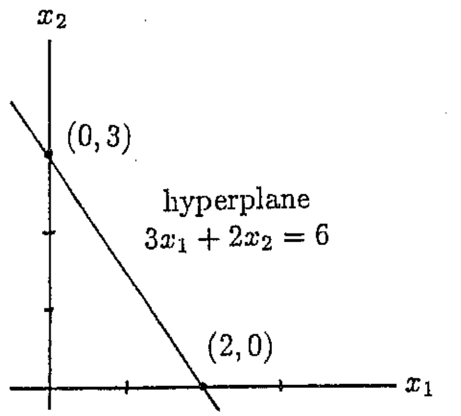


FIGURE 2

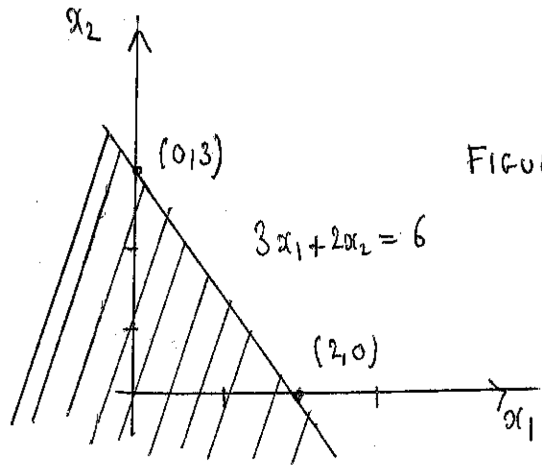


FIGURE 3

Next we want to find the side of this line that satisfies the given inequality. We will use $(0,0)$ as our test point. Since $3(0) + 2(0) < 6$ is true, we want the side that contains the test point $(0,0)$. For convenience in problems that have more than one inequality we will *shade the part we* want. SEE FIGURE 3.

When we want to find the solution set to a system of inequalities we must find the intersection of the solution sets of the various inequalities.

EXAMPLE 2: Geometrically (graphically) indicate the solution set of:

$$2x_1 + 3x_2 \geq 5$$
$$-x_1 + x_2 \leq 0$$

Solution:

Step 1. The solution set of the first inequality consists of the hyperplane whose equation is $2x_1 + 3x_2 = 5$ and the half space given by $2x_1 + 3x_2 > 5$. The hyperplane has intercepts at $(0, \frac{5}{3})$ and $(\frac{5}{2}, 0)$. We draw the line through these two points. $(0,0)$ is not on this line so we may use it as a test point. Since $2(0) + 3(0) > 5$ is false, we do not want the side of the hyperplane that contains the origin. We SHADE THE OTHER HALF, FIGURE 4.

Step 2. The hyperplane of the second constraint is the line whose equation is $-x_1 + x_2 = 0$. This line has only one intercept: $(0,0)$. To get a second point on the line we will (arbitrarily) let $x_1 = 2$. Then $x_2 = 2$ also. We plot the two points $(0,0)$ and $(2,2)$ and then draw the line through them. To find the half-space we need a test point. Since the origin $(0,0)$ is on the hyperplane, it cannot be the test point. We will use the point $(2,0)$. We want the side that contains $(2,0)$ because $-(2) + (0) < 0$ is true. We shade THIS HALF. See FIGURE 5.

Step 3. The solution set we really want is shown in FIGURE 6. It arises from carrying out Steps 1 and 2 on the same set of axes. (That will be our normal practice.) The solution consists of the shaded region together with the parts of the hyperplanes that form the "edges" of the shaded region.

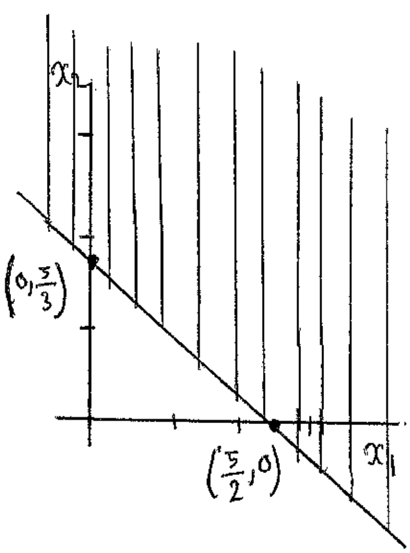


FIGURE 4

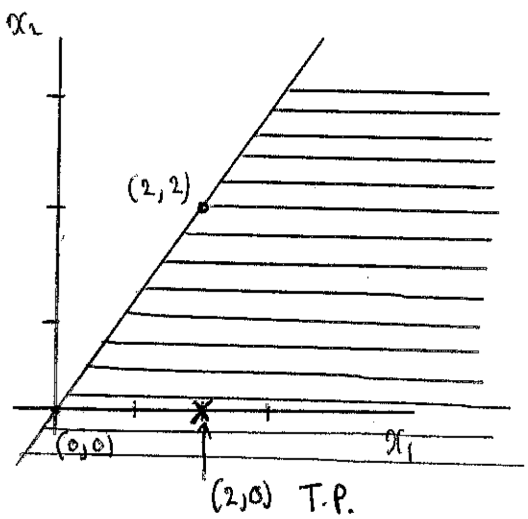


FIGURE 5

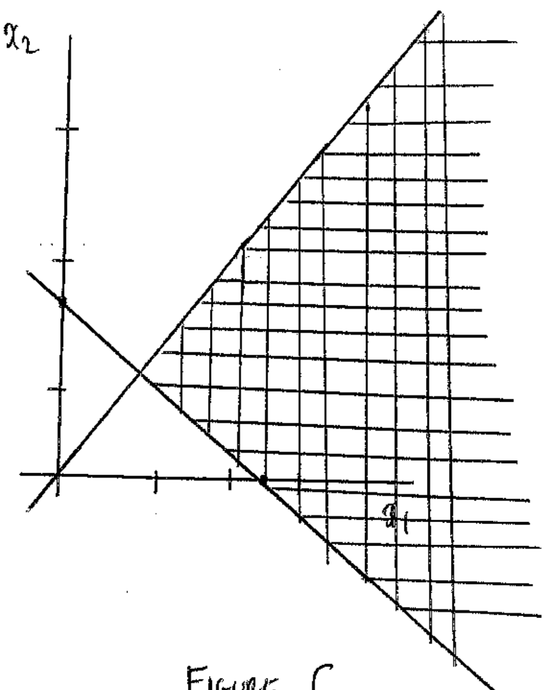


FIGURE 6

FEASIBLE REGION SHADED.

EXAMPLE 3:

SAME 2 CONSTRAINTS AS EXAMPLE 2 WITH NON-NEGATIVITY CONSTRAINTS ADDED.

$$2x_1 + 3x_2 \geq 5$$
$$-x_1 + x_2 \leq 0$$
$$x_1 \geq 0 \quad x_2 \geq 0$$

Solution:

SHADING THE AREA DETERMINED BY $x_1 \geq 0$, $x_2 \geq 0$ GIVES THE FIRST QUADRANT: SEE FIGURE 7. THE FINAL SHADED REGION IS SHOWN IN FIGURE 8.

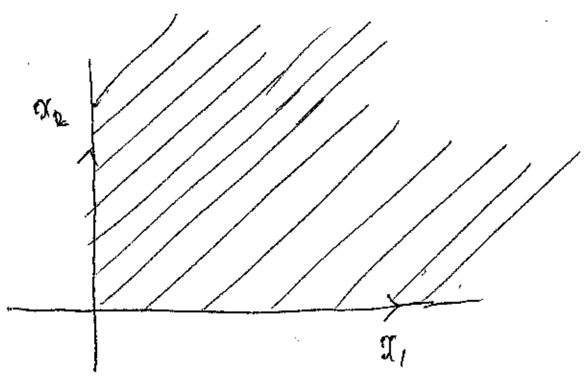


FIGURE 7

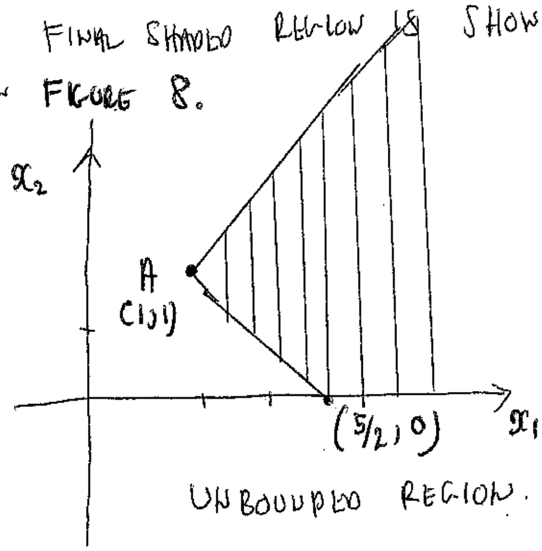


FIGURE 8

HOW DO WE FIND THE COORDINATES OF POINT A IN FIGURE 8?

WE USE ELIMINATION TO SOLVE THE SYSTEM:

$$2x_1 + 3x_2 = 5$$
$$-x_1 + x_2 = 0$$

THE SOLUTION IS $x_1 = 1$, $x_2 = 1$ ~ $(x_1, x_2) = (1, 1)$.

(HERE WE COULD USE ROW REDUCTION, BUT ELIMINATION IS QUICKER. WE USE ROW REDUCTION IN LP4, LP5.)

EXAMPLE 4: Sketch the feasible region for the LP model:

$$\begin{array}{ll} \max & P = 4x_1 + x_2 \\ \text{if} & 2x_1 + x_2 \leq 3 \\ & x_1 + 3x_2 \leq 6 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{array}$$

Solution: PRACTICE THIS YOURSELF. LABEL THE 4 CORNER POINTS WITH THEIR COORDINATES. SEE FIGURE 9.

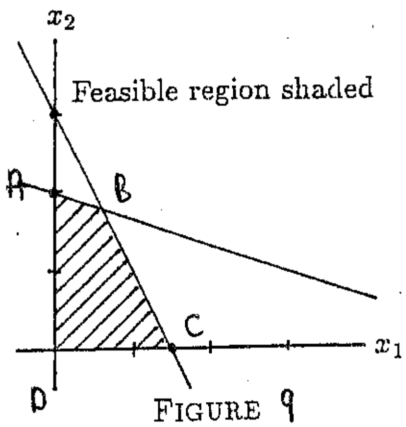


FIGURE 9

$A = (0, 2)$

$B = (3/5, 9/5)$

$C = (3/2, 0)$

$D = (0, 0)$

CORNER POINTS COORDINATES.

NOTICE THAT WHEN SHADING THE FEASIBLE REGION WE DO NOT USE THE OBJECTIVE FUNCTION P.

In order to actually solve an LP problem geometrically we must introduce and use the objective function. We do this by graphing several level lines. The **level line with value** v for an objective function of the form $z = ax_1 + bx_2$ is the line (hyperplane) whose equation is $ax_1 + bx_2 = v$. Different values of v yield different level lines, but they will all be parallel to each other.

EXAMPLE 5: Draw the level lines for $P = 2x_1 + 4x_2$ for the values 10, 4, and 0.

Solution: The three level lines have equations $2x_1 + 4x_2 = 10$, $2x_1 + 4x_2 = 4$, and $2x_1 + 4x_2 = 0$. They are graphed in FIGURE 10.

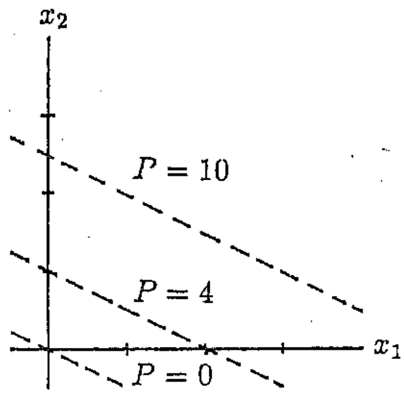


FIGURE 10

We want to emphasize that every point (r, s) on the level line with value v will yield the value v when put into the objective function. The level line with value v shows all the places where the objective function has value v. For example, in Example 5 above, $(1, 2)$ lives on the level line $P = 10$ because it yields 10 when put into the objective function: $P = 2(1) + 4(2) = 10$.

Before we solve an LP problem geometrically, let us make sure exactly what we want to accomplish:

1. To solve a max LP problem we must find a point in the feasible region which, when put into the objective function, gives a value which is as large as, or larger than, the value given by any other feasible point. *Our solution consists of the coordinates of this point and the value of the objective there.*
2. To solve a min LP problem we must find a point in the feasible region which, when put into the objective function, gives a value which is as small as, or smaller than, the value given by any other feasible point. *Our solution consists of the coordinates of this point and the value of the objective there.*

USING LEVEL LINES WE SEE THAT THE SOLUTION OF AN LP PROBLEM OFTEN OCCURS AT A CORNER POINT (VERTEX) OF THE FEASIBLE REGION.

WE ARE READY TO SOLVE OUR FIRST LP PROBLEM; WHICH IS THE PROBLEM FROM EXAMPLE 4.

EXAMPLE 6: SOLVE THE LP PROBLEM

$$\begin{array}{ll} \max & P = 4x_1 + x_2 \\ \text{if} & 2x_1 + x_2 \le 3 \\ & x_1 + 3x_2 \le 6 \\ & x_1 \ge 0, \quad x_2 \ge 0 \end{array}$$

BY THE GEOMETRIC METHOD.

Solution: FIND + SHADE FEASIBLE REGION AND LABEL CORNER POINTS AS BEFORE:

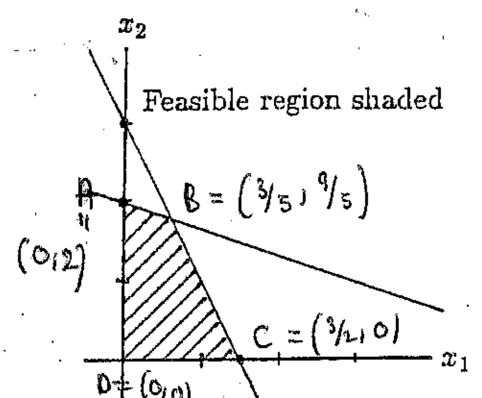


FIGURE 11

Corner point		P	(C FOR EXAMPLE 10).
A =	(0, 2)	2	
B =	(3/5, 9/5)	4 1/5	
C =	(3/2, 0)	6	
D =	(0, 0)	0	

THEN MAKE A TABLE AND EVALUATE THE OBJECTIVE FUNCTION P AT THE 4 POINTS A, B, C, D. CHOOSE THE MAXIMUM (max) IN P COLUMN.

SOLN: $(3/2, 0)$, $P = 6$.

EXAMPLE 7: CHANGE max P IN EXAMPLE 6 TO min C.

$$\begin{array}{ll} \min & C = 4x_1 + x_2 \\ \text{if} & 2x_1 + x_2 \le 3 \\ & x_1 + 3x_2 \le 6 \\ & x_1 \ge 0, \quad x_2 \ge 0 \end{array}$$

Solution: THE FEASIBLE REGION + TABLE ARE THE SAME. IT IS A min C PROBLEM SO WE CHOOSE THE MINIMUM IN C COLUMN: SOLN: $(0,0)$, $C=0$

UNBOUNDED FEASIBLE REGION.

EXAMPLE 8: Solve the following LP problem graphically.

$$\begin{aligned} \min \quad & C = x_1 + x_2 \\ \text{if} \quad & 2x_1 + x_2 \geq 4 \\ & x_1 + 2x_2 \geq 4 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{aligned}$$

Solution: CHECK THE FOLLOWING FEASIBLE REGION AND TABLE. NOTE THAT THE FEASIBLE REGION IS UNBOUNDED.

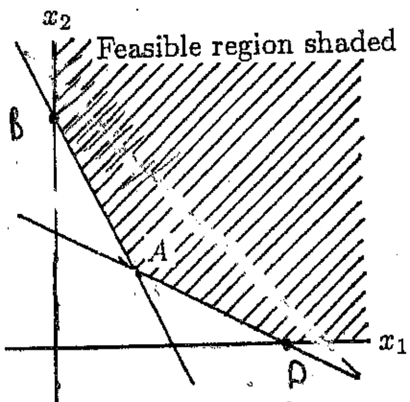


FIGURE 12

Corner point	C
$A = (4/3, 4/3)$	$8/3$
$B = (0, 4)$	4
$D = (4, 0)$	4

SOLN: $(4/3, 4/3)$, $C = 8/3$

EXAMPLE 9: For the feasible region of Example 8, find $\max P = x_1 + x_2$.

Solution: HERE NOTE THE FOLLOWING TABLE:

Feasible point	P
(10, 10)	20
(100, 100)	200
(1000, 1000)	2000
⋮	⋮

IN THIS CASE THE OBJECTIVE FUNCTION P CAN GET ARBITRARILY LARGE — THERE IS NO 'MAXIMUM' OR 'LARGEST' VALUE OF P.

THIS PROBLEM HAS AN UNBOUNDED MAXIMUM.

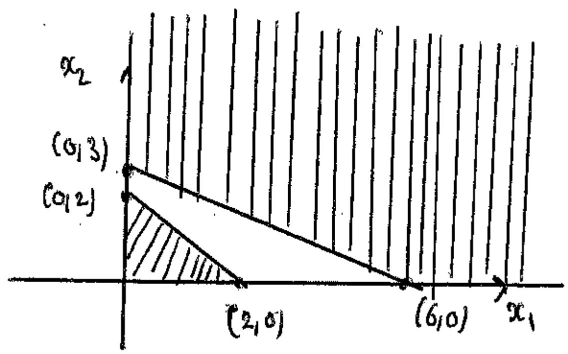
EMPTY FEASIBLE REGION

It is possible that the feasible region for a LP problem is empty; that is, there are no points which satisfy the constraints. In this case there will be no solution to the problem. This happens in the following problem:

EXAMPLE 10:

$$\begin{array}{ll} \max & P = x_1 + x_2 \\ \text{if} & x_1 + 2x_2 \geq 6 \\ & x_1 + x_2 \leq 2 \\ & x_1 \geq 0, \quad x_2 \geq 0 \end{array}$$

Solution:



FEASIBLE REGION IS EMPTY: NOTHING IS SHADED TWICE. NO SOLN TO LP PROBLEM.

WE SUMMARIZE THE 3 CASES WHICH (FOR US) CAN OCCUR.

CONSIDER A LP PROBLEM WITH FEASIBLE REGION FR, AND OBJECTIVE Z.

a) IF FR IS BOUNDED THEN Z HAS BOTH A MAXIMUM AND MINIMUM VALUE ON FR. CHECK VERTICES FOR THE max OR min.

eg EXAMPLES 6,7.

b) IF FR IS UNBOUNDED:

IF IT IS A min C PROBLEM, THERE IS A SOLN. CHECK VERTICES.

eg Secs vs. Clerks or EXAMPLE 8

IF IT IS A max P PROBLEM, THERE IS NO SOLN: UNBOUNDED MAX. P CAN BE MADE ARBITRARILY LARGE, eg EXAMPLE 9.

c) FR IS EMPTY. THE LP PROBLEM HAS NO SOLN. eg EXAMPLE 10.